Homework 2

Question 1

Solve the following differential equations using classical methods. Assume zero initial conditions.

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 20x = 5u(t)$$

Repeat the question using Laplace transform, assuming zero initial conditions.

Solution:

In this case the characteristic equation is $\lambda^2 + 6\,\lambda + 20 = 0$ with solutions $\lambda_{1,2} = -3 \pm j3.3166$

so the homogeneous solution is $x_h(t) = Ae^{(-3+j3.3166)t} + Be^{(-3-j3.3166)t}$.

The postulated particular solution is $x_p(t) = C$.

Substituting the particular solution into the original differential equation one gets

$C = 0.25$.

So, we have $x(t) = 0.25 + Ae^{(-3+j3.3166)t} + Be^{(-3-j3.3166)t}$.

To find A and B note that $x(0) = 0.25 + A + B = 0$.

Also, since $x'(t) = (-3 + j3.3166)Ae^{(-3+j3.3166)t} + (-3 - j3.3166)Be^{(-3-j3.3166)t}$, and $x'(0) = (-3 + j3.3166)A + (-3 - j3.3166)B = 0$.

Solving simultaneously for A and B gives $A = -0.125 + j0.1131 = B^*$.

Therefore $x(t) = 0.25 + (-0.125 + j0.1131)e^{(-3+j3.3166)t} + (-0.125 - j0.1131)e^{(-3-j3.3166)t}$.

This expression can be simplified into $x(t) = 0.25 - e^{-3t}[0.25\cos 3.3166t + 0.2262\sin 3.3166t]$.

The Laplace transform of the differential equation, assuming zero initial conditions, is

$$((s^2 + 6s + 20)X(s) = \frac{5}{s}$$

Solving for $X(s)$ and expanding by partial fractions,

$$X(s) = \frac{5}{s(s^2 + 6s + 20)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 6s + 20}$$

Multiplying by the lowest common denominator and equating the same powers of s on both sides,

$A + B = 0, \quad 6A + C = 0, \quad 20A = 5$

Combining equations,

$$A = \frac{1}{4}, \;\; B = -\frac{1}{4}, \;\; C = -\frac{3}{2}$$

Thus,

$$X(s) = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s + \frac{3}{2}}{s^2 + 6s + 20}$$

The roots of the quadratic are complex and located at -3 ± 3.317

Thus, use the following form for exponentially damped sinusoids.

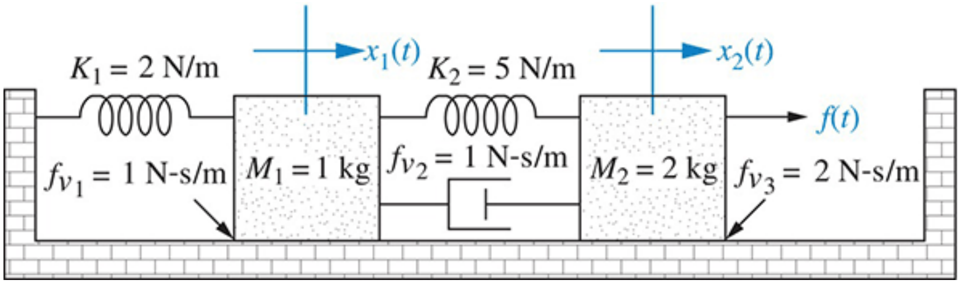
$$X(s) = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}(s+3) + \frac{3}{4\sqrt{11}}\sqrt{11}}{(s+3)^2 + 11}$$

Taking the inverse Laplace transform,

$$x(t) = 0.25 - e^{-3t}(0.25\cos 3.317t + \sin 3.317t)$$

Question 2

Find the transfer function, $G(s) = X_2(s)/F(s)$, for the system shown below:



Solution:

The system has two independent translational displacements, so we can write the following two equations:

$$\begin{aligned} X_1: &\quad (s^2 + 2s + 7)X_1(s) - (s + 5)X_2(s) = 0 \\ X_2: &\quad -(s + 5)X_1(s) + (2s^2 + 3s + 5)X_2(s) = F(s) \end{aligned}$$

Solving we get:

$$X_2(s) = \frac{\begin{vmatrix} s^2 + 2s + 7 & 0 \\ -(s+5) & F(s) \end{vmatrix}}{\begin{vmatrix} s^2 + 2s + 7 & -(s+5) \\ -(s+5) & 2s^2 + 3s + 5 \end{vmatrix}} = \frac{(s^2 + 2s + 7)F(s)}{(s^2 + 2s + 7)(2s^2 + 3s + 5) - (s + 5)^2}$$

$$= \frac{(s^2 + 2s + 7)F(s)}{2s^4 + 7s^3 + 24s^2 + 21s + 10}$$

The resulting transfer function can be written as

$$\frac{X_2(s)}{F(s)} = \frac{1}{2}\frac{s^2 + 2s + 7}{s^4 + 3.5s^3 + 12s^2 + 10.5s + 5}$$

Question 3

Represent the following transfer function in state space.

$$T(s) = \frac{s(s+2)}{(s+1)(s^2+2s+5)}$$

$$T(s) = \frac{s(s+2)}{(s+1)(s^2+2s+5)} \Rightarrow$$

$$T(s) = \frac{s^2+2s+0}{s^3+3s^2+7s+5}$$

state-space model:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -7 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$$

$$D = 0$$

Questions 4

Find the transfer function $G(s) = \frac{Y(s)}{R(s)}$ for the following system represented in state space:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0] x$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \quad C = [1 \ 0 \ 0] \quad D = 0.$$

The transfer function is

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - A)^{-1} \cdot B + D$$

Using the Matlab to address it.

```
syms s;
A=[0 1 0; 0 0 1; -3 -2 -5];
B=[0;0;10];
C=[1 0 0];
D=[0];
G=C*inv(s*eye(3,3)-A)*B+D
```

G =

$$\frac{10}{s^3 + 5s^2 + 2s + 3}$$

$$G(s) = \frac{10}{s^3 + 5s^2 + 2s + 3}$$